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Axisymmetric Harmonic Infrapolynomials in R^N

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1. INTRODUCTION

During the last two decades we have witnessed an intensive development of the subject of infrapolynomials on sets $\omega \subset \mathbf{C}$. As we recall, an infrapolynomial on ω is a polynomial $p \in \mathbf{P} = \{\zeta^n + \sum_1^n a_k \zeta^{n-k}\}$ such that there exists no other $q \in \mathbf{P}$ for which $q(\zeta) = 0$ for $\zeta \in \omega' = \{\zeta \in \omega : p(\zeta) = 0\}$ and $|q(\zeta)| < |p(\zeta)|$ for $\zeta \in \omega - \omega'$. A leader in this development was Professor Walsh, the man whom we are honoring and of whom I was privileged to be the first Ph. D. student.

In the present paper we attempt a parallel development for harmonic infrapolynomials on three-dimensional sets. Our results will be expressed in terms of three coordinate systems in $\mathbf{R}^3$: rectangular (x, y, z); cylindrical (x, ρ, ϕ) with

$$\rho^2 = y^2 + z^2, \qquad y = \rho \cos \phi, \qquad z = \rho \sin \phi;$$

and spherical (r, θ, ϕ) with

$$x = r \cos \theta, \qquad \rho = r \sin \theta. \tag{1.1}$$

By an *axisymmetric function* in $\mathbf{R}^3$ we mean one that is independent of ϕ; that is, a function which assumes the same value at all points of the circle $x = x_0$, $\rho = \rho_0$ [abbreviated: circle (x_0, ρ_0)]. As the domain of such a function, we take an *axisymmetric* set Ω in $\mathbf{R}^3$; that is a set such that, if point $(x_0, \rho_0, \phi_0) \in \Omega$, also point $(x_0, \rho, \phi) \in \Omega$ for all ρ and ϕ, $0 \leqslant \rho \leqslant \rho_0$ and $0 \leqslant \phi \leqslant 2\pi$. Thus an axisymmetric set Ω may consist of points on the x-axis,

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circular disks having their centers on the x-axis and their planes perpendicular to the x-axis, and the interiors of surfaces of revolutions which are cut in a single circle by any plane perpendicular to the x-asis. The meridian section $\omega \subset \mathbf{C}$ of Ω is an *axiconvex* region, meaning that $\zeta \in \omega$ implies $\lambda\zeta + (1 - \lambda)\bar{\zeta} \in \omega$ for all real λ, $0 \leqslant \lambda \leqslant 1$.

Let us first consider axisymmetric harmonic polynomials $H(x, \rho)$ of degree n. As is well known [5, p. 254], every such polynomial can be written in the form

$$H(x, \rho) = \sum_{j=0}^{n} a_j r^j P_j(x/r), \tag{1.2}$$

where $P_j(u)$ is the Legendre polynomial of degree j.

Of special importance is the class

$$\mathbf{H} = \{H(x, \rho) \colon a_n = 1\} \tag{1.3}$$

of axisymmetric harmonic polynomials with leading coefficient of one and therefore strictly of degree n. Let us compare two polynomials $P(x, \rho) \in \mathbf{H}$ and $Q(x, \rho) \in \mathbf{H}$ on a given axisymmetric set Ω in $\mathbf{R}^3$ relative to some suitably defined norm $\| H(x, \rho)\|$. We say that Q is an underpolynomial of P on Ω if

$$\| Q(x, \rho)\| < \| P(x, \rho)\| \tag{1.4}$$

for all circles $(x, \rho) \subset \Omega$. Let us denote by $U(P, \Omega)$ the class of all underpolynomials of P on Ω. If $U(P, \Omega) = \varnothing$ for some $P \in \mathbf{H}$, we say that P is an *axisymmetric harmonic infrapolynomial on* Ω.

In the sequel we shall investigate the properties of the class $\mathbf{I}(\Omega)$ of axisymmetric harmonic infrapolynomials on a given axisymmetric set Ω. We shall determine some conditions on $P \in \mathbf{H}$ in order that $P \in \mathbf{I}(\Omega)$ and also determine the location of the zeros of all $P \in \mathbf{I}(\Omega)$ in relation to the set Ω. In order to do this, we shall bring together the methods of two hitherto disjoint disciplines the theory of infrapolynomials on sets $\omega \subset \mathbf{C}$ and the theory of a certain integral operator, whose development is largely due to Professor Stefan Bergman; see [1 and 2].

2. Integral Representations for $H(x, \rho)$ and $\| H(x, \rho)\|$

Let us define as the *associate* of $H(x, \rho)$ the polynomial

$$h(\zeta) = \sum_{k=0}^{n} a_k \zeta^k, \qquad a_n = 1, \quad \zeta \in \mathbf{C}. \tag{2.1}$$

Thus ζ^n is the associate of $r^n P_n(x/r)$.

In view of the Whittaker formula [4, p. 312–315],

$$r^k P_k(x/r) = (1/2\pi) \int_0^{2\pi} (x + i\rho \cos t)^k \, dt, \tag{2.2}$$

we have the result:

THEOREM 2.1. *Let $H(x, \rho)$ be an axisymmetric harmonic polynomial and let $h(\zeta)$ be its associate.*
Then

$$H(x, \rho) = (1/2\pi) \int_0^{2\pi} h(x + i\rho \cos t) \, dt. \tag{2.3}$$

More generally, if $f(\zeta)$ is analytic in a region ω which is the meridian section of an axisymmetric region Ω, then

$$F(x, \rho) = (1/2\pi) \int_0^{2\pi} f(x + i\rho \cos t) \, dt \tag{2.4}$$

satisfies Laplace's equation $\nabla^2 F = 0$ and so is an axisymmetric harmonic function in Ω. In fact, (2.3) and (2.4) are special cases of the operator introduced by Bergman [1, p. 43]:

$$F(x, y, z) = (1/2\pi i) \int_{|\tau|=1} f(\zeta, \tau) \, \tau^{-1} \, d\tau, \tag{2.5}$$

acting upon the function $f(\zeta, \tau)$ that is analytic in ζ on some region in $\mathbf{C}$ and continuous in τ for $|\tau| = 1$. On setting

$$\zeta = x + (1/2)(yi + z)\,\tau + (1/2)(yi - z)\,\tau^{-1}, \tag{2.6}$$

the operator transforms $f(\zeta, \tau)$ into the function $F(x, y, z)$ which together with $\mathscr{R}F(x, y, z)$ and $\mathscr{I}F(x, y, z)$ is harmonic in a certain region of $\mathbf{R}^3$.

In view of the integral representation (2.3) for an axisymmetric harmonic polynomials $H(x, \rho)$, it is natural to define the norm $\| H(x, \rho)\|$ of $H(x, \rho)$ by the formula

$$\| H(x, \rho)\|^2 = (1/2\pi) \int_0^{2\pi} | h(x + i\rho \cos t)|^2 \, d\sigma(t). \tag{2.7}$$

Here and in the subsequent formulas, $\sigma(t)$ denotes a monotonically increasing function for $0 \leqslant t \leqslant 2\pi$. In the special case $\sigma(t) \equiv t$, we denote as norm $\| H(x, \rho)\|_t$. Thus,

$$\| r^n P_n(x/r)\|^2 = (1/2\pi) \int_0^{2\pi} (x^2 + \rho^2 \cos^2 t)^n \, d\sigma(t).$$

More generally, using (1.2), we may expand (2.7) as a hermitian form in the a_k, the coefficients of which form are homogeneous polynomials in x and ρ.

Let us now consider the harmonic polynomial $H(x, \rho)$ which has for its associate

$$h(\zeta) = p(\zeta)\, q(\zeta), \tag{2.8}$$

where p and q are, respectively, polynomials of degrees k and $n - k$. Let us denote by $P(x, \rho)$ and $Q(x, \rho)$ the axisymmetric harmonic polynomials which have $p(\zeta)$ and $q(\zeta)$, respectively, as associates. To indicate a kind of factor relation among $H(x, \rho)$, $P(x, \rho)$ and $Q(x, \rho)$, we follow Bergman in defining the operation

$$P(x, \rho)^* \, Q(x, \rho) = (1/2\pi) \int_0^{2\pi} p(x + i\rho \cos t)\, q(x + i\rho \cos t)\, d\sigma(t). \tag{2.9}$$

Thus, whereas the product $P(x, \rho)\, Q(x, \rho)$ is not ordinarily harmonic, the product $P(x, \rho)^* \, Q(x, \rho)$ is harmonic and so the operation converts the family of axisymmetric harmonic polynomials into an algebra.

Obviously we may express the norm of any axisymmetric harmonic polynomial $H(x, \rho)$ in terms of the product in (2.9), as follows.

THEOREM 2.2. *If $H(x, \rho)$ is any axisymmetric harmonic polynomial, its norm $\| H(x, \rho)\|$ as defined by* (2.7) *satisfies the relation*

$$\| H(x, \rho)\|^2 = H(x, \rho)^* \, \overline{H(x, \rho)}. \tag{2.10}$$

The product $P(x, \rho)^* \, \overline{Q(x, \rho)}$ is in general not a harmonic function but serves the purpose of "inner vector product" in the space of axisymmetric harmonic function.

We now prove the following theorem.

THEOREM 2.3. *Let $P(x, \rho)$ and $Q(x, \rho)$ be any two axisymmetric harmonic polynomials. Then*

$$| P(x, \rho)^* \, Q(x, \rho)| \leqslant \| P(x, \rho)\| \, \| Q(x, \rho)\|. \tag{2.11}$$

Proof. Using (2.9) and Schwarz inequality, we infer that

$$\begin{aligned} | P(x, \rho)^* \, Q(x, \rho)| &\leqslant (1/2\pi) \int_0^{2\pi} | p(x + i\rho \cos t \,||\, q(x + i\rho \cos t)|\, d\sigma \\ &\leqslant \left\{(1/2\pi) \int_0^{2\pi} | p(x + i\rho \cos t\, |^2\, d\sigma\right\}^{1/2} \\ &\quad \times \left\{(1/2\pi) \int_0^{2\pi} | q(x + i\rho \cos t\, |^2\, d\sigma\right\}^{1/2}. \end{aligned}$$

That is, (2.11) is valid for all (x, ρ).

If we choose $Q(x, \rho) = 1$ and $\sigma(t) = t$ in Theorem 2.3, we obtain the following result.

COROLLARY 2.1. *If $H(x, \rho)$ is an axisymmetric harmonic polynomial, then for all (x, ρ)*

$$| H(x, \rho)| \leqslant \| H(x, \rho)\|_t . \tag{2.12}$$

As may be seen from (2.7), the equality sign holds in both (2.11) and (2.12) when $\rho = 0$ and when P, Q and H are each constants, but does not seem to hold in any other case.

3. STRUCTURE OF AXISYMMETRIC HARMONIC INFRAPOLYNOMIALS

We shall now use well-known theorems about the structure of infrapolynomials on $\omega \subset \mathbf{C}$ in order to get some corresponding results regarding axisymmetric harmonic infrapolynomials on $\Omega \subset \mathbf{R}^3$. It will be helpful first to prove the following.

THEOREM 3.1. *Let $P(x, \rho) \in \mathbf{H}$, the class of axisymmetric harmonic polynomials defined by* (1.3). *Let $\omega \subset \mathbf{C}$ be a bounded axiconvex region and $\Omega \subset \mathbf{R}^3$ be the axisymmetric region having ω as its meridian section. If $P(x, \rho)$ is an infrapolynomial on the closure $\bar{\Omega}$ of Ω, then its associate $p(\zeta)$ is an infrapolynomial on the closure $\bar{\omega}$ of ω.*

Proof. If the contrary were true, p would have an underpolynomial q on ω; that is,

$$| q(\zeta)| = | p(\zeta)| \qquad \text{for} \quad \zeta \in \omega' = \{\zeta \in \bar{\omega} \colon p(\zeta) = 0\}, \tag{3.1}$$

$$| q(\zeta)| < | p(\zeta)| \qquad \text{for} \quad \zeta \in \bar{\omega} - \omega'. \tag{3.2}$$

Let $q(\zeta)$ be the associate of $Q(x, \rho)$. Clearly, $Q(x, \rho) \in \mathbf{H}$ and

$$\begin{aligned} \| Q(x, \rho)\|^2 &= (1/2\pi) \int_0^{2\pi} | q(x + i\rho \cos t)|^2 \, d\sigma \\ &< (1/2\pi) \int_0^{2\pi} | p(x + i \cos t)|^2 \, d\sigma = \| P(x, \rho)\|^2. \end{aligned}$$

Hence, $P(x, \rho)$ would have an underpolynomial $Q(x, \rho)$ on $\bar{\Omega}$, contradicting the hypothesis that $P(x, \rho)$ is an infrapolynomial on $\bar{\Omega}$.

For example, since in $\mathbf{C}$ ζ^n is an infrapolynomial on the unit disk $| \zeta | \leqslant 1$, we infer that, in $\mathbf{R}^3$, $r^n P_n(x/r)$ is an infrapolynomial on the unit ball $x^2 + \rho^2 \leqslant 1$.

We now propose to use Theorem 3.1 in conjunction with the following well-known result due to Fekete [3, pp. 15–19].

THEOREM 3.2. *Let E, a closed bounded set in $\mathbf{C}$ containing at least $n+1$ points, have an infrapolynomial $p(\zeta)$, with $p(\zeta) \neq 0$ for $\zeta \in E$. Then there exist an integer m with $n \leqslant m \leqslant 2n$, a set of $m+1$ constants $\lambda_j > 0$ with $\lambda_0 + \lambda_1 + \cdots + \lambda_m = 1$ and a set of $m+1$ points $\{\zeta_0, \zeta_1, \ldots, \zeta_m\} \subset E$ such that $p(\zeta)$ is a factor of the polynomial*

$$f(\zeta) = \sum_{k=0}^{m} \lambda_k \psi_k(\zeta) \tag{3.3}$$

where

$$\psi_k(\zeta) = (\zeta - \zeta_0) \cdots (\zeta - \zeta_{k-1})(\zeta - \zeta_{k+1}) \cdots (\zeta - \zeta_m). \tag{3.4}$$

In applying Theorem 3.2, we need to choose E to be a bounded axiconvex region ω.

For, the integration in (2.3) requires that, if point $x + i\rho \in \omega$, then also point $x + i\rho \cos t \in \omega$ for $0 \leqslant t \leqslant 2\pi$. In view of Theorems (3.1) and (3.2), we are led now to the following theorem.

THEOREM 3.3. *Let Ω be a bounded axisymmetric region in $\mathbf{R}^3$ and let $P(x, \rho)$ be an n-th degree axisymmetric harmonic infrapolynomial on the closure of Ω. Then there exist an integer m, $n \leqslant m \leqslant 2n$, a set of $m+1$ constants $\lambda_j > 0$ with $\lambda_0 + \lambda_1 + \cdots + \lambda_m = 1$, a set of circles*

$$(x_0, \rho_0; x_1, \rho_1; \ldots; x_m, \rho_m\} \subset \Omega,$$

and an axisymmetric harmonic polynomial $G(x, \rho)$ of degree $m - n$ such that

$$\sum_{k=0}^{m} \lambda_k \Psi_k(x, \rho) = P(x, \rho)^* G(x, \rho), \tag{3.5}$$

where $\Psi_k(x, \rho)$ is the axisymmetric harmonic polynomial

$$\Psi_k(x, \rho) = -(1/2\pi)(\partial/\partial x_k) \int_0^{2\pi} \prod_{j=0}^{m} (x + i\rho \cos t - x_j - i\rho_j)\, dt. \tag{3.6}$$

Proof. Since $P(x, \rho)$ is an infrapolynomial on Ω, its associate $p(\zeta)$ is by Theorem 3.1 an infrapolynomial on ω, the meridian section of Ω. By Theorem 3.2, there is a polynomial $g(\zeta)$ of degree $m - n$ such that

$f(\zeta) = p(\zeta)\, g(\zeta)$. Hence $f(\zeta)$ is the associate of $P(x, \rho)^* G(x, \rho)$, where $G(x, \rho)$ is the axisymmetric harmonic polynomial having $g(\zeta)$ as associate. On the ohter hand, $f(\zeta)$ may be written in the form (3.3) and so we are led to (3.5).

4. Null Circles of Axisymmetric Harmonic Infrapolynomials

By a *null circle* (x_0, ρ_0) of an axisymmetric harmonic polynomial $H(x, \rho)$ we mean that $H(x_0, \rho_0) = 0$. A null circle is therefore the intersection of the level surfaces $S_1 : \mathscr{R}H(x, \rho) = 0$ and $S_2 : \mathscr{I}H(x, \rho) = 0$. The set of all null circles of $H(x, \rho)$ is finite unless $\mathscr{R}H(x, \rho) \equiv 0$ or $\mathscr{I}H(x, \rho) \equiv 0$ when it is the entire level surface S_2 or S_1, respectively.

Let us first recall the following result about the zeros of an infrapolynomial on $\omega \subset \mathbf{C}$, due to Féjer [3, p. 23].

Theorem 4.1. *Let E be a closed bounded set in $\mathbf{C}$ and $p(\zeta)$ an infrapolynomial on E. Then all the zeros of p lie in the convex hull of E.*

In applying Theorem 4.1, we must again replace E by a bounded axiconvex region ω which is the meridian section of an axisymmetric region Ω. Let us denote by c_1 and c_2 the two points which are on the real axis, left and right of ω, respectively, and from which ω subtends an angle of π/n. Thus, ω lies in the intersection of the two sectors

$$-(\pi/2n) \leqslant \arg(\zeta - c_1) \leqslant (\pi/2n), \tag{4.1}$$

$$\pi - (\pi/2n) \leqslant \arg(\zeta - c_2) \leqslant \pi + (\pi/2n). \tag{4.2}$$

Let us denote by $K_1(\omega, n)$ and $K_2(\omega, n)$ the cones obtained on revolving about the axis of reals the two sectors

$$\pi - (\pi/2/n) \leqslant \arg(\zeta - c_1) \leqslant \pi + (\pi/2n) \tag{4.3}$$

$$-\pi/2n \leqslant \arg(\zeta - c_2) \leqslant \pi/2_n\,. \tag{4.4}$$

Alternatively, to obtain, for example, $K_2(\omega, n)$ geometrically, we may take a double nappe cone of vertex angle π/n and slide it as far as possible to the left with its axis along the x-axis and yet have the left nappe contain Ω. The right nappe is then $K_2(\omega, n)$.

We are now in a position to establish

Theorem 4.2. *Let $\omega \subset \mathbf{C}$ be a bounded axiconvex region, which is the meridian section of Ω, an axisymmetric region in $\mathbf{R}^3$. Let $P(x, \rho)$ be an axi-*

symmetric harmonic infrapolynomial on Ω. Then no circle (x, ρ) for which $P(x, \rho) = 0$ may lie in either cone:

$$K_j(\omega, n): \qquad 0 < \rho \leqslant (-1)^j(x - c_j)\tan(\pi/2n), \qquad j = 1, 2, \tag{4.5}$$

where the c_j are defined as above.

Proof. By Theorem 3.1, the associate $p(\zeta)$ of $P(x, \rho)$ is an infrapolynomial on ω and by Theorem 4.1 all the zeros ζ_j, $j = 1, 2,..., n$ of $p(\zeta)$ lie in the convex hull κ of ω. The region κ also lies in the intersection of the two sectors (4.1) and (4.1).

Let us write

$$p(\zeta) = (\zeta - \zeta_1)(\zeta - \zeta_2) \cdots (\zeta - \zeta_n)$$

and thus

$$P(x, \rho) = (1/2\pi) \int_0^{2\pi} \prod_{j=1}^{n} (x + i\rho\cos t - \zeta_j)\, dt.$$

Let us assume that there is a circle (x_0, ρ_0) in the cone $K_2(\omega, n)$ such that $P(x_0, \rho_0) = 0$. Then

$$\int_0^{2\pi} w(t)\, dt = 0, \tag{4.5}$$

where

$$w(t) = \prod_{j=1}^{n} (\zeta_j - x_0 - i\rho_0 \cos t). \tag{4.6}$$

These assumptions require point $x_0 + i\rho_0$ to lie in sector (4.4) and therefore point $x_0 + i\rho_0 \cos t$ also to lie interior to sector (4.4) for all t, $0 \leqslant t \leqslant 2\pi$. Since $\zeta_j \in \kappa$ for $j = 1, 2,..., n$ and since κ lies in sector (4.2), it follows that

$$\pi - (\pi/2n) < \arg(\zeta_j - x_0 - i\rho_0 \cos t) < \pi + (\pi/2n)$$

for each j and, because of (4.6),

$$n\pi - (\pi/2) < \arg w(t) < n\pi + (\pi/2).$$

Hence, $\mathscr{R}[e^{-n\pi i}w(t)] > 0$ for $0 \leqslant t \leqslant 2\pi$ and thus $\mathscr{R}[e^{-\pi i} \int_0^{2\pi} w(t)\, dt] > 0$. This contradicts (4.5) and thus the assumption that $P(x_0, \rho_0) = 0$ for circle $(x_0, \rho_0) \subset K_2(\omega, n)$, is invalid. Using similar reasoning for circles $(x_0, \rho_0) \subset K_1(\omega, n)$, we complete the proof of Theorem 4.2.

Remark 1. Theorem 4.2 remains valid if $\| H(x, \rho)\|$ as defined by (2.7) is replaced by any other norm for which Theorem 3.1 is true.

5. Generalization to $\mathbf{R}^N$

We shall extend the preceding results to axisymmetric harmonic functions $F(x_1, x_2, ..., x_N)$ of N real variables; that is, solutions of the Laplace equation

$$\Delta^2 F = \sum_{j=1}^{N} (\partial^2 F/\partial x_j^2) = 0. \tag{5.1}$$

The axisymmetric case corresponds to the one in which F is a function just of x and ρ where

$$x = x_1, \qquad \rho^2 = x_2^2 + x_3^2 + \cdots + x_N^2. \tag{5.2}$$

In this case (5.1) reduces to

$$(\partial/\partial x)(\rho^{N-2}\, \partial F/\partial x) + (\partial/\partial \rho)(\rho^{N-2}\, \partial F/\partial \rho) = 0. \tag{5.3}$$

On introducing polar coordinates into Eq. (5.3)

$$x = r \cos \theta, \qquad \rho = r \sin \theta$$

and using the method of separating variables, we find the basic solutions of (5.3) in the form

$$r^n P_n^{(\mu)}(\cos \theta), \qquad 2\mu = N - 2, \tag{5.4}$$

where $P_n^{(\mu)}(\cos \theta) = P_n^{(\alpha,\alpha)}(\cos \theta)$, $2\alpha = N - 3$, and where $P_n^{(\alpha,\beta)}(\cos \theta)$ and $P_n^{(\mu)}(\cos \theta)$ are, respectively, the Jacobi and Gegenbauer polynomials of degree n.

We are thus led to consider the axisymmetric harmonic polynomials in $\mathbf{R}^N$

$$H(x, \rho) = \sum_{j=0}^{n} A_j r^j P_j^{(\mu)}(x/r), \qquad 2\mu = N - 2, \tag{5.5}$$

with $A_n = 1$. For such a polynomial the following holds.

THEOREM 5.1. *The harmonic function* (5.5) *may be written in the form*

$$H(x, \rho) = 2^{3-N} \Gamma(\mu)^{-2} \int_0^{\pi} h(x + i\rho \cos t) \sin^{N-3} t \, dt \tag{5.6}$$

$$h(\zeta) = \sum_{j=0}^{n} a_j \zeta^j, \tag{5.7}$$

with $a_j = [\Gamma(j + 2\mu)/j!]\, A_j$.

Proof. The expression (5.5) follows directly from the representation [2, p. 167]

$$r^n P_n^{(\mu)}(\cos\theta) = \frac{2^{1-2\mu}\Gamma(n+2\mu)}{n!\,\Gamma(\mu)^2}\int_0^\pi (x+i\rho\cos t)^n \sin^{N-3} t\,dt. \quad (5.8)$$

We refer to the polynomial in (5.7) as the *associate* of the polynomial $H(x, \rho)$ given by (5.5).

By analogy with Section 2, we define the norm $\| H(x,\rho)\|$ by the expression

$$\| H(x,\rho)\|^2 = 2^{3-N}\Gamma(\mu)^{-2}\int_0^\pi | h(x+i\rho\cos t)|^2 \sin^{N-3} t\, d\sigma(t). \quad (5.9)$$

If now we are given two polynomials $P(x, \rho)$ and $Q(x, \rho)$ of type (5.5), we say that $Q(x, \rho)$ is an underpolynomial of $P(x, \rho)$ on an axisymmetric region $\Omega \subset \mathbf{R}^N$ if $\| Q(x,\rho)\| < \| P(x,\rho)\|$ for all $(x,\rho)\in\Omega$ and that $P(x, \rho)$ is an axisymmetric harmonic infrapolynomial on $\bar\Omega$ if it has no underpolynomial $Q(x, \rho)$ on $\bar\Omega$.

By the same reasoning as for Theorem 3.1, we may establish the following:

THEOREM 5.2. *Let $\omega \subset \mathbf{C}$ be a bounded axiconvex region and let $\Omega \subset \mathbf{R}^N$ be the region comprising the loci $x = x_0$, $x_2^2 + x_3^2 + \cdots + x_N^2 = \rho_0^2$ for all $x_0 + i\rho_0 \in \omega$. If $P(x, \rho)$ is an axisymmetric harmonic infrapolynomial on $\bar\Omega$, its associate $p(\zeta)$ is an infrapolynomial on $\bar\omega$.*

Again, since (5.6) differs from (2.3) principally because of the nonnegative factors $2^{3-N}\Gamma(\mu)^{-2}\sin^{N-3}t$ in (5.5), we may use the same reasoning as for Theorem 4.2 to show the following theorem to be valid.

THEOREM 5.3. *Let $\omega \subset \mathbf{C}$ be a bounded axiconvex in C and let $\Omega \subset \mathbf{R}^N$ be the region comprising the loci $x_1 = x_0$, $x_2^2 + x_3^2 + \cdots + x_N^2 = \rho_0^2$ for all $x_0 + i\rho_0 \in \omega$. If $P(x, \rho)$ is an axisymmetric harmonic infrapolynomial on $\bar\Omega$, then no locus (x_0, ρ_0) for which $P(x, \rho) = 0$ has points in either of the cones*

$$0 < (x_2^2 + x_3^2 + \cdots + x_N^2)^{1/2} \leqslant \pm(x_1 - c_j)\tan(\pi/2n) \quad (5.10)$$

where the c_j are defined as for Theorem 4.2. *for $j = 1, 2$.*

Also results analogous to Theorems 2.3 and 3.3 are valid, but their statement and proof are left to the reader.

6. Extension to Certain other Harmonic Infrapolynomials in $\mathbf{R}^3$

Let us finally consider harmonic polynomials of the form

$$F(x, y, z) = \sum_{j=J}^{n} A_j r^j P_j^{m(n-j)}(x/r) \cos m(n-j)\phi, \tag{6.1}$$

where $A_n = 1$; m and J are integers with $m > 0$, $J \geqslant [mn/(m+1)]$, and $P_j^k(\cos\theta)$ is the "associated Legendre function" [4, p. 323]. Clearly, $F(x, y, z)$ is a harmonic polynomial, but not ordinarily axisymmetric. We may show that a representation of $F(x, y, z)$ in the form (2.5) is possible on choosing as associate

$$f(\zeta, \tau) = \tau^{-mn} f_0(\tau^m \zeta), \tag{6.2}$$

where

$$f_0(\zeta) = \sum_{j=J}^{n} a_j \zeta^j = \zeta^J f_1(\zeta) \tag{6.3}$$

with

$$a_j = [j + m(n-j)]!/j!]\, A_j$$

and ζ given by (2.6) or, since $\tau = e^{ti}$, equivalently, by

$$\zeta = x + i(y \cos t + z \sin t) = x + i\rho \cos(t - \phi). \tag{6.4}$$

We may deduce the desired relation directly from the formula [4, p. 326]

$$r^n P_n{}^k(\cos\theta) = \frac{(j+k)!}{j!(2\pi)} \int_0^{2\pi} (x + i\rho \cos t)^n e^{-kti}\, dt.$$

That is,

$$F(x, y, z) = (1/2\pi) \int_0^{2\pi} f(x + i\rho \cos(t - \varphi), e^{ti})\, dt. \tag{6.5}$$

We next define the norm $\| F(x, y, z)\|$ in terms of the variables x, y, z or x, ρ, φ in such a way that the norm has the integral representation

$$\| F(x, y, z)\|^2 = (1/2\pi) \int_0^{2\pi} | f(x + i\rho \cos(t - \varphi), e^{ti})|^2\, d\sigma(t). \tag{6.6}$$

We then say that $F(x, y, z)$ is an infrapolynomial on a given region $\Omega \subset \mathbf{R}^3$ if no polynomial $G(x, y, z)$ of the same type as (6.1) exists such that $\| G(x, y, z)\| < \| F(x, y, z)\|$ for all $(x, y, z) \in \Omega$.

By reasoning similar to that in the proof of Theorem 3.1, we can now establish the following.

THEOREM 6.1. *Let $\omega \subset \mathbf{C}$ be a bounded axiconvex region and let $\Omega \subset \mathbf{R}^3$ be the axisymmetric region whose meridian cross section is ω. If the harmonic polynomial $F(x, y, z)$ given by* (6.1) *is an infrapolynomial on Ω, then the corresponding polynomial $f(\zeta)$ in* (6.3) *is an infrapolynomial on ω.*

According to Theorem 4.1, the zeros of the infrapolynomial

$$f_1(\zeta) = \prod_{j=1}^{n-J} (\zeta - \zeta_j)$$

lie in the convex hull κ of ω. Accordingly, since

$$\begin{aligned} f(\zeta, \tau) &= e^{-mnti} e^{Jmti} \zeta^J \prod_{j=1}^{n-J} (e^{mti}\zeta - \zeta_j) \\ f(\zeta, \tau) &= \zeta^J \prod_{j=1}^{n-J} (\zeta - \zeta_j e^{-mti}) \end{aligned} \tag{6.8}$$

the zeros $\zeta_j e^{-mti}$ therefore lie in the disk $|\zeta| \leqslant \delta$, where $\delta = \max |\zeta|$ for $\zeta \in \omega$.

If now $F(x, y, z)$ is an infrapolynomial on Ω and if $F(x_0, y_0, z_0) = 0$, then according to (6.5) and (6.8),

$$\int_0^{2\pi} w(t)\, dt = 0,$$

where

$$w(t) = (0 - x_0 - i\rho_0 \cos(t - \varphi_0))^J \prod_{j=1}^{n-J} [\zeta_j e^{-mti} - x_0 - i\rho_0 \cos(t - \varphi_0)].$$

From here on, the reasoning is similar to that for Theorem 4.2. We thus arrive at the following result.

THEOREM 6.2. *Let $\Omega \subset \mathbf{R}^3$ be an axisymmetric region whose meridian section is a bounded axiconvex region ω.*

Let $\delta = \max |\zeta|$ for $\zeta \in \omega$. If $F(x, y, z)$ given by (6.1) *is a harmonic infrapolynomial on Ω, then no point (x, y, z) for which $F(x, y, z) = 0$ lies in either of the cones:*

$$0 < \rho \leqslant \pm x \tan(\pi/2n) - \delta \sec(\pi/2n). \tag{6.9}$$

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